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Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in reversible systems

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Abstract

The persistence from a singular limit is studied of homoclinic orbits to a saddle-centre equilibrium in four-dimensional reversible vector fields. The linearisation is assumed to have eigenvalues $\pm i\omega$, $\pm\lambda$, $\lambda, \omega > 0$, with the singular limit being $\lambda \rightarrow 0$. Recent work by E. Lombardi has shown the generic *non-persistence* of such homoclinic solutions under perturbations that break the integrability of a normal form due to a splitting term (Melnikov function) that is exponentially small in λ/ω .

The arguments of Lombardi are adapted to show that given the correct sign of the coefficient b_2 in the normal form that couples the hyperbolic to the elliptic dynamics, then one should expect persistence in a *codimension-one* sense. That is, keeping nonlinear terms fixed but treating λ and ω as independent parameters, all terms in the leading order expression for the Melnikov function has a countable sequence of sign changes as a function of ω . A zero of this function is argued to imply an ω -value at which a curve of homoclinic orbits bifurcates from $\lambda = 0$ in the parameter plane. The large- ω asymptotics of this sequence of bifurcation points is determined solely by the value of b_2 .

This approximate analytical result is supported by careful numerical experiments on a class of fourth-order reversible equations with arbitrary quadratic nonlinear terms. The implications of these results are discussed for the existence of solitary water waves of elevation in the presence of surface tension and for recently discovered *embedded solitons* in a variety of (non-integrable) nonlinear optics models.

1 Introduction

In recent years there has been a wide interest in the application of so-called exponential asymptotics to study the solutions of ordinary differential equations (ODEs). A particularly challenging problem from the theory of singular perturbations is that of ‘fast oscillations’ where an autonomous vector field whose dynamics vary on an $O(1)$ -timescale is perturbed by a time-periodic function varying on an $O(1/\varepsilon)$. An exemplar of such class of problems is the rapidly forced pendulum

$$\ddot{x} + \sin x = \rho \sin(t/\varepsilon),$$

where ρ and ε are both small; see Holmes, Marsden & Scheurle (1988), Delshams & Seara (1992), Ellison, Kummer & Sáenz (1993), Gelfreich (1994), Fontich (1995), Fiedler & Scheurle (1996). A key question for such systems is to determine the effect of the forcing on the separatrix trajectory of the unforced problem and, if there are transverse homoclinic points the associated Poincaré map, to work out the size in parameter space of the ensuing chaos. For such problems the Melnikov function, that is the splitting distance of the stable and unstable manifolds of the hyperbolic equilibrium, turns out to be exponentially small in ε and must in general be calculated via the study of its singularities in the complex time domain.

This article concerns a related problem where a planar vector field with an $O(\varepsilon)$ timescale is coupled to another planar vector field whose (uncoupled) dynamics consists of periodic orbits varying on an

$O(1)$ timescale. The problem comes about through the study a codimension-one bifurcation for four-dimensional reversible systems (typically also Hamiltonian, though not necessarily) . Such systems often arise as travelling wave or steady state reductions of partial differential equations on the real line which model various phenomena in mechanics, fluids and optics (see references in Champneys (1998, 1999) and Sections 1.2 and 1.3 below). In such problems, solutions which decay to zero at $\pm\infty$ are of particular interest as they represent localised modes or solitary waves. This interest then leads to the question of existence of homoclinic orbits to the ODE system. (Unless explicitly stated otherwise, in what follows ‘homoclinic’ will always be taken to mean ‘homoclinic to the zero solution’.)

1.1 Fourth-order reversible systems

The class of systems to be studied are four-dimensional reversible systems

$$\dot{x} = f(x), \quad x \in \mathbb{R}^4, \quad Rf(x) = -f(Rx), \quad R^2 = \text{Id}, \quad \mathcal{S} = \text{fix}(R) \cong \mathbb{R}^2, \quad (1.1)$$

where R is a linear transformation such that $x = 0$ is a *saddle centre equilibrium* that is

$$f(0) = 0, \quad \sigma(Df(0)) = \{\pm\lambda \pm i\omega\} \quad \text{for some } \omega > 0, \lambda \geq 0 \quad (1.2)$$

For simplicity we will suppose that f is an analytic function of x and of whatever parameters it may depend on. We shall be specifically interested in the limit $\lambda \rightarrow 0$, which may be termed a degenerate saddle centre bifurcation. We shall focus on the case where the normal form of the singular saddle-centre bifurcation truncated at finite order contains a homoclinic orbit in the hyperbolic part whose amplitude is $O(\lambda)$ as $\lambda \rightarrow 0$ (a so-called *type-I* normal form, see Section 2 below). The key issue is to determine what is the effect of the normal-form breaking terms that effectively couple the separate hyperbolic and elliptic linear dynamics.

In a series of recent papers Lombardi (1996, 1997b, 1999, 1997a) has studied such problems via careful estimation of the singularities of the ensuing Melnikov integral in complex time (Section 2 below is devoted to a summary of some of these results, although not the method). He shows that the contribution of the normal-form-breaking coupling terms is to make the Melnikov integral an exponentially small function of λ , much in the same way as for the rapidly forced pendulum. Provided the coefficient of the exponentially small term is non-zero, which Lombardi (1999) shows to be *generically* the case, then it will break the connection between the stable and unstable manifolds of the origin. Hence, generically, for small positive λ there is no homoclinic orbit to the origin.

In contrast, Lombardi (1996) shows that, for all small $\lambda > 0$, there is a one-parameter family of homoclinic connections to *each member* of the one-parameter family of periodic orbits that form the centre manifold of the origin, provided the amplitude of the periodic orbit is bigger than some minimum amplitude. This minimum amplitude is essentially given by the *same* exponentially small function of α that governs the non-existence of homoclinic-to-zero solutions.

The purpose of the present paper is to show that the coefficient of these exponentially small terms may well undergo a sequence of zeros as another parameter is varied. Importantly, reversible homoclinic orbits to saddle-centre equilibria in four-dimensional reversible systems are of codimension-one. Hence it only makes sense to study the question of their persistence from the degenerate limit with *two* parameters being active. Hence, in what follows we shall consistently treat models in which λ and ω are *two independent parameters* assuming that the nonlinear terms remain fixed. From this point of view we revisit Lombardi’s analysis and come up with the startling conclusion, at least formally, that given just a checkable sign condition of a certain normal form coefficient then there are *an infinite number of ω -values* which correspond to the vanishing of the Melnikov function and hence the persistence of homoclinic orbits for small λ . That is, there are a sequence of ‘bifurcation points’ from $\lambda = 0$ at which curves of homoclinic orbits emanate into the positive λ component of the (λ, ω) -plane. See Figure 1 below in which the parameter $-a$ is like λ and $-b$ like ω .

Our aim is to show how this result explains some earlier reported numerical observations on a variety of model systems. So, before presenting detailed findings, let us first recall some of the numerical motivation.

1.2 5th-order KdV models for water waves

The so-called 5th-order Korteweg–de Vries (KdV) equation is a first-order partial differential equation (PDE) model for waves on water of finite depth under the influence of gravity and surface tension (Zufiria 1987, Hunter & Scheurle 1988). It may be written as

$$v_t = v_{zzzzz} - bv_{zzz} + 2vv_z. \quad (1.3)$$

This equation was previously also used to study wave propagation in various branches of mathematical physics (Kakutani & Ono 1969, Hasimoto 1970, Kawahara 1972, Ostrovsky, Gorshkov, Papko & Pikovsky 1979). One finds solitary-wave solutions of (1.3) by setting $v(z, t) = y(x = z - at)$ and noting that the equation then becomes a derivative with respect to x . For solitary waves we require zero background disturbance at $x = \pm\infty$, therefore after choosing the constant of integration to be zero, one seeks homoclinic-to-zero solutions to the ODE

$$y'''' - by'' + ay + y^2 = 0 \quad (1.4)$$

where a prime denotes differentiation with respect to x . Viewed as a four-dimensional dynamical system, with two real parameters a and b , (1.4) is reversible under

$$R : (y, y', y'', y''') \rightarrow (y, -y', y'', -y'''). \quad (1.5)$$

(It is also conservative, and can be written in Hamiltonian co-ordinates, but that will not concern us in what follows.) Equation (1.4) is in fact the simplest example of the more general class of reversible systems considered in Section 3 below. These systems are equivalent to ODEs of the form

$$y'''' - by'' + ay + g(y, y', y'', y''') = 0, \quad (1.6)$$

where the nonlinearity g is such that the system is still reversible under (1.5).

Many things are known about existence of (infinitely many distinct) homoclinic solutions to (1.4) for $a > 0$ when the origin is a hyperbolic equilibrium (see Champneys (1998) for a review). This paper considers the complementary region $a < 0$ where the origin is a saddle centre. In particular, we shall also assume $b < 0$ so that a singular saddle-centre occurs as $a \rightarrow 0$. The corresponding limit has proved of great interest in the full Euler-equation formulation of the water-wave problem, of which (1.3) is just a model (Beale 1991, Iooss & Kirchgassner 1992, Sun & Shen 1993, Lombardi 1997b, Sun 1999). In fact the parameters a and b are chosen so that they mimic the dimensionless parameters of the full model, b is like Bond number minus $1/3$, and a is like Froude number minus 1. It might be tempting to speculate whether that the analysis of the present paper has anything to say about this full water wave problem which is an infinite-dimensional reversible dynamical system (Mielke 1991). In what follows however, we shall concentrate solely on four-dimensional systems.

Note now that, owing to its pure power nonlinearity, (1.4) with $a < 0$, $b < 0$ can be rescaled to include just a single parameter

$$\varepsilon y'''' + y'' - y + y^2 = 0, \quad (1.7)$$

which is the form studied by a number of authors (Amick & Kirchgassner 1989, Hammersley & Mazzarino 1989, Amick & McLeod 1991). When $\varepsilon = 0$ (equivalent to a blow up of (1.4) at $a = 0$), (1.7) admits the homoclinic solution $y(x) = (3/2)\text{sech}(x/2)$, and the question is whether a branch of such solutions survives for $\varepsilon > 0$. The answer is no. Specifically Amick & McLeod (1991) proved that there are no reversible homoclinic solutions to the origin of (1.7) for any ε , and no non-reversible ones

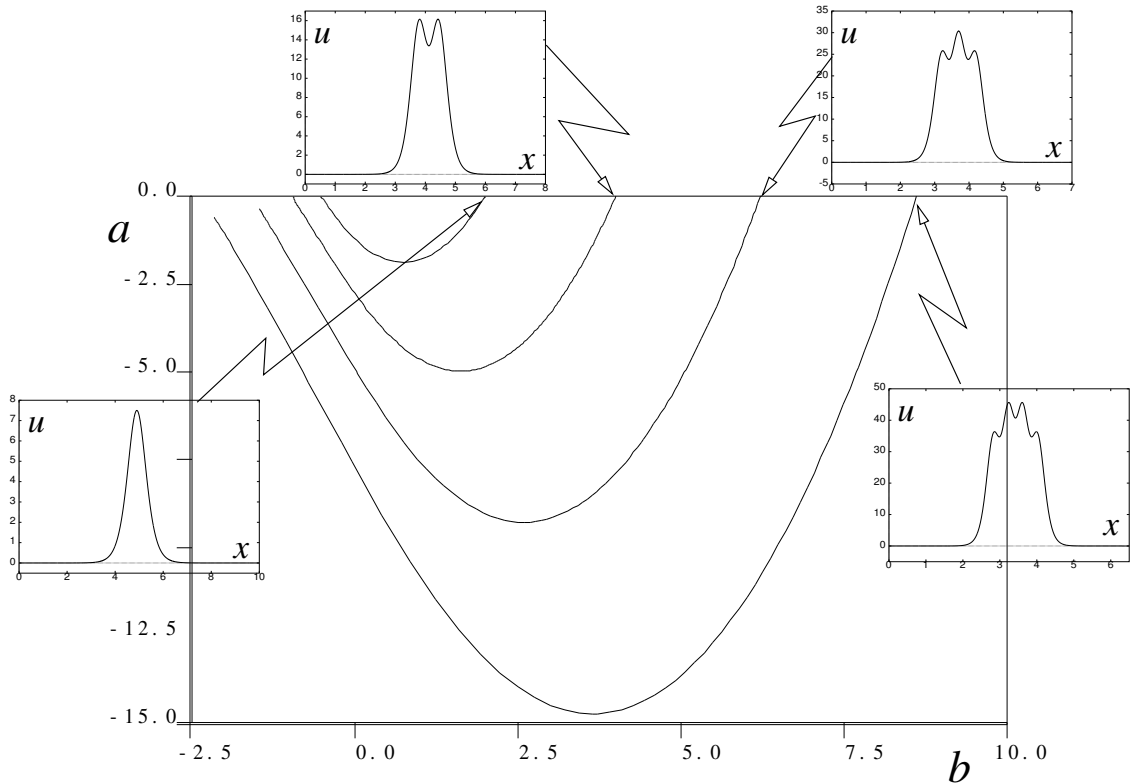


Figure 1: Numerically obtained homoclinic solutions to (1.8) (after (Champneys & Groves 1997)). Note the numerical difficulties in continuing the solutions right up to $a = 0$ for $b < 0$ (in which limit their amplitudes tend to zero). This is due to the exponential smallness of the splitting as $a \rightarrow 0$.

for ε sufficiently small. However it is known that there are homoclinic solutions to periodic orbits, both with a single large peak (Pomeau, Ramani & Grammaticos 1988, Amick & Toland 1992, Boyd 1991, Grimshaw & Joshi 1995, Sun 1998) and with multiple peaks (Champneys & Lord 1997, Calvo & Akylas 1997). Moreover, the minimum amplitude of periodic orbit to which a homoclinic orbit exists scales like $(1/\varepsilon) \exp(-K/\sqrt{\varepsilon})$ for some $K > 0$.

In view of the analysis to come, it is not a surprise that (1.4) does not have persistence of homoclinic orbits, since it can be scaled to a one parameter problem. In order to study a genuinely two-parameter model, Champneys & Groves (1997) considered the equation

$$(2/15)y'''' - by'' + ay + (3/2)y^2 - (1/2)(y')^2 + (yy')' = 0, \quad (1.8)$$

which is derived by taking higher-order terms into account in the water-wave derivation of the 5th-order KdV equation (Kichenassamy & Olver 1996, Craig & Groves 1994). Note that (1.8) can be put in the form (1.6) by a scaling of x and redefinition of b in order to remove the coefficient $(2/15)$ on the highest derivative). Champneys & Groves (1997) found numerically that for a sequence of negative b -values there is indeed persistence of homoclinic orbits from the singular limit $a = 0$ along bifurcating codimension-one lines, see Figure 1. In fact, the first of these bifurcating solutions has the explicit form

$$u(t) = 3 \left(b + \frac{1}{2} \right) \operatorname{sech}^2 \left(\sqrt{\frac{3(2b+1)}{4}} t \right), \quad a = \frac{3}{5}(2b+1)(b-2), \quad b \geq -1/2.$$

Note that the term ‘bifurcation’ is not used in a strict mathematical sense here, since formally to all orders of a as $a \rightarrow 0^-$ homoclinic solutions exist for all $b < 0$ (Kichenassamy & Olver 1996). So

we have a ‘beyond all orders’ bifurcation. In order to avoid confusion, in what follows we shall refer to this phenomenon as *persistence* from the singular limit. It is this apparent regular sequence of b -values at which persistence occurs that the analysis below shall seek to explain. Note that there is also numerical evidence (Kivshar, Champneys, Cai & Bishop 1998) of similar persistence for (1.6) with $g = y^3 + (3/4)y(yy'' + y'^2) = 0$ which is akin to (1.8) but with odd symmetry.

Before leaving water-wave models, we should also note the results by Grimshaw & Cook (1996) on a model consisting of a coupled pair of (usual, 3rd-order) KdV-equations. They used exponential asymptotics (as in Grimshaw & Joshi (1995)) in the neighbourhood of a singular saddle-centre to calculate the minimum amplitude of periodic orbits to which homoclinic orbits may be formed. They found zeros of the coefficient of this exponentially small term along a regular sequence of codimension-one lines which, it would be reasonable to conjecture, correspond to where homoclinic connections to zero occur.

1.3 ‘Embedded solitons’ in nonlinear optics

Recently a number of systems of equations of the form (1.1) arising in nonlinear optics have been analysed. For example Champneys, Malomed & Friedman (1998) studied an ODE of the form

$$DU'' + iU' + \chi U + U|U|^2 + U^* = 0, \quad (1.9)$$

which arises as a reduction of a system of PDEs describing propagation of optical pulses in a fibre equipped with a so-called Bragg grating. By taking the real and imaginary parts of U and U' as dynamical variables, (1.9) is equivalent to a fourth-order reversible system with reversibility acting by complex conjugation. Note it also has odd symmetry $U \rightarrow -U$. The dispersion term D may be thought of as a singular perturbation modelling the higher-order effects of wave propagation, and χ is the carrier wave number of the optical pulse. The equilibrium at $U = 0$ is a saddle centre (1.2) for $|\chi| < 1$, $D \neq 0$ and undergoes a singular saddle-center bifurcation at $\chi = 1$. In this limit, numerical evidence (Champneys et al. 1998) suggests that there are precisely three values of $D > 0$ at which there is persistence of a homoclinic orbit from the singular limit.

Also, Yang, Malomed & Kaup (1999) considered the system

$$\begin{aligned} (1/2)U'' - kU + U^*V + \Gamma(|U|^2 + 2|V|^2)U &= 0, \\ -(1/2)\delta V'' + (q - 2k)V + (1/2)U^2 + 2\Gamma(|V|^2 + 2|U|^2)V &= 0, \end{aligned} \quad (1.10)$$

which models pulse propagation or localised modes of a planar wave guide in an optical medium with competing quadratic and cubic nonlinearities. For this fourth-order reversible system, the saddle-centre parameter region is given by $0 < k < q/2$ with the singular limit occurring at $k = 0$. For fixed $\delta > 0$ and $\Gamma < 0$ (see also Champneys, Malomed, Yang & Kaup (1999) for similar results for $\delta < 0$, $\Gamma > 0$) there is numerical evidence of at least two homoclinic solutions which persist from the singular limit (an exhaustive numerical search has yet to be performed). Yang et al. (1999) gave such homoclinic solutions the name *embedded solitons* (‘soliton’ in the optics community refers to any pulse solution, and does not imply integrability). This name arises because, when interpreted in terms of the coupled nonlinear Schrödinger PDE system from which (1.10) arise, the fact that the origin is a saddle centre means that the pulse solutions are embedded into the continuous spectrum of radiation modes. One key feature from this work appears to be that the simplest branch of embedded solitons, with no internal oscillations (cf. the first bifurcating solution in Figure 1), is linearly neutrally stable as a solution of the PDE system.

Embedded solitons have also been found in another model for optical media with quadratic nonlinearities, solitary wave solutions of which solve the ODE system (Champneys & Malomed 1999a)

$$\begin{aligned} -ku_1 + iu_1' + u_1^* + u_3u_1 &= 0, \\ -(4k + q)u_3 + Du_3'' + u_1u_1^* &= 0. \end{aligned} \quad (1.11)$$

Taking u_3 (assumed to be real) and $\text{Re}(u_1)$, $\text{Im}(u_1)$ as dynamical variables this system has the form (1.1). The saddle centre region for $D > 0$ is given by $|k| < 1$, $4k + q < 0$, with the singular limit occurring at $|k| = 1$. Strong numerical evidence was found that there are a countable infinity of codimension-one branches of homoclinic solutions in this region. However, *none* of them are born in persistence from the singular limit. Instead, all the branches terminate at the codimension-two point $k = -1$, $q = 1/4$ where all four eigenvalues of the linearized problem are zero.

1.4 Outline

The preceding two sections gave numerical examples from the recent literature that suggest there may be countably many persistence points of homoclinic orbits to singular saddle centres for two-parameter, four-dimensional reversible systems; there may be finitely many; or there may be none at all. The rest of the paper is devoted to motivating why this is so. Specifically, Section 2 summarises in a non-technical way the results of Lombardi concerning *non-persistence* of homoclinic orbits in the normal form for singular saddle-centres. We shall highlight a particular case of the analysis where certain remainder terms can be argued to be small. In that case, a reinterpretation of the results provides strong evidence of persistence in the codimension-one sense depicted in Figure 1, given a simple sign condition. Section 3 will then go on to study numerically a class of systems which can be expressed in the form (1.6) with g composed of arbitrary reversible quadratic terms. The main results are summarised in Table 1 below. Good agreement with the theory is found for most combinations of quadratic terms. Finally Section 4 draws conclusions, shows how different results may arise for example systems with degenerate normal forms, such as those with odd symmetry, and suggests directions for future research.

2 Persistence via normal form theory

Consider a general, analytic fourth-order reversible system (1.1) satisfying (1.2).

First let us see why, for $\lambda > 0$, homoclinic orbits to the origin of such systems are of codimension-one in the parameter space and they should be reversible; see also Devaney (1976), Mielke, Holmes & O'Reilly (1992), Koltsova & Lerman (1995), Champneys (1998). Note that the origin has a one-dimensional unstable manifold $W^u(0)$ and a one-dimensional stable manifold $W^s(0)$. In order for a solution $\gamma(t)$ homoclinic to the origin to exist, then one of the two branches of $W^u(0)$ must coincide with a branch of $W^s(0)$. For non-reversible, non-conservative systems (but for which the linearisation (1.2) remains true) such identification of two lines in \mathbb{R}^4 would be of codimension three. For reversible systems though, special conditions apply. Firstly, the linearisation (1.2) is generic. Secondly, there are two distinct kinds of homoclinic orbits: either the orbit is itself reversible, in which case $\gamma(0) \in \text{fix}(R)$; or γ does not intersect the symmetric section $\text{fix}(R)$, and so $R\gamma(-t)$ is a distinct homoclinic orbit. In the former case, since we require only a point intersection between the one-dimensional $W^u(0)$ and the two-dimensional $\text{fix}(R)$, this is clearly of codimension one among the class of reversible systems. For the latter case, since each homoclinic orbit never sees $\text{fix}(R)$, the codimension is the same as for a non-reversible system; i.e. three. Hence, non-reversible homoclinic orbits to saddle-centres are extremely rare, whereas a curve of reversible ones could be described by two active system parameters. Hence, we shall focus only on reversible homoclinic orbits in what follows.

It is worth mentioning, since most physical examples of reversible systems are also Hamiltonian, that the existence of a first integral H changes the above argument, since the effective phase space is a three-dimensional level set of H . Simple counting of dimensions now shows that reversible homoclinic orbits are of codimension-one for conservative reversible systems also, but that non-reversible orbits are of codimension two. So, in either conservative systems or not, reversible homoclinic orbits are the most likely and if they occur they should do so along lines in a parameter plane.

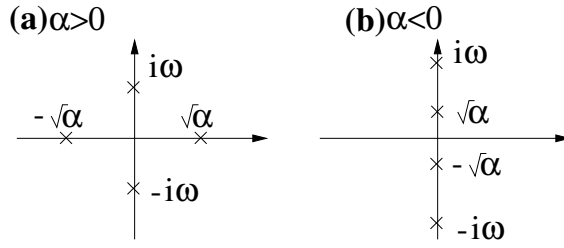


Figure 2: *Eigenvalues for the normal form (2.2)*

2.1 Normal form for type-I reversible singular-saddle-centre bifurcations

Suppose $\lambda = 0$ in (1.1), (1.2) and that this double eigenvalue of $Df(0)$ is not semi-simple (i.e. we are in the generic case of a Jordan block in the linearisation). Then a straightforward calculation (Lombardi 1996, Lemma 2.1) shows that after a change of co-ordinates we may assume generically that the linearisation is of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & \omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

and that the reversibility R acts in either of two ways,

$$\text{either } R : (x_2, x_4) \rightarrow -(x_2, x_4) \quad (\text{type I}), \quad \text{or } R : (x_1, x_4) \rightarrow -(x_1, x_4) \quad (\text{type II}). \quad (2.1)$$

The distinction between the two cases is thus whether the unique eigenvector associated with the zero eigenvalue is itself symmetric under R (type II) or anti-symmetric (type I). It is worth remarking that all known examples of such systems derived from physical models appear to be of type I. This is because systems from applications are almost invariably also Hamiltonian, and reversibility acts on the momentum variables only. The linear subsystems associated with the vanishing of eigenvalues $\pm\sqrt{\alpha}$ for such systems naturally take the form $\ddot{q} = \alpha q$ where q is a co-ordinate whose momentum is related to \dot{q} . This is type-I behaviour. For this reason, like Lombardi, we shall focus exclusively on type-I systems.

As always, the linearisation at the equilibrium and its symmetry properties (in this case a reversibility) determine the resonant terms in a normal form up to any order. Applying standard normal form techniques (e.g. Kuznetsov (1995)) at $\lambda = 0$ one arrives for type-I reversibility at the following system (Lombardi 1996):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= P(x_1, x_3^2 + x_4^2, \alpha, \omega), \\ \dot{x}_3 &= -x_4 Q(x_1, x_3^2 + x_4^2, \alpha, \omega), \\ \dot{x}_4 &= x_3 Q(x_1, x_3^2 + x_4^2, \alpha, \omega), \end{aligned} \quad (2.2)$$

up to remainder terms beyond the order of the normal-form. Here P and Q are arbitrary polynomials which may be truncated at any finite order to produce a reduced-order normal form, and $\alpha = \lambda^2$. Observe that the normal form (2.2) is completely integrable with invariants

$$K = \sqrt{x_3^2 + x_4^2} \quad \text{and} \quad J = x_2^2 - 2 \int_{x_1} P(w, K^2, \alpha, \omega) dw. \quad (2.3)$$

We next need to make an assumption on how the parameter α perturbs the linearisation. Without loss of generality, provided α is not a bad parameter, we can suppose that $\alpha > 0$ creates two real eigenvalues $\pm\lambda = \pm\sqrt{\alpha}$, and that $\alpha < 0$ creates two imaginary eigenvalues $\pm i\sqrt{-\alpha}$, see Figure 2.

Therefore, after a scaling of x_1, \dots, x_4 if necessary, the full system can be assumed to read

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \alpha x_1 - (3/2)x_1^2 - b_1(\omega)(x_3^2 + x_4^2) + N_2(x; \alpha, \omega), \\ \dot{x}_3 &= -x_4(\omega + b_2(\omega)x_1) + N_3(x; \alpha, \omega), \\ \dot{x}_4 &= x_3(\omega + b_2(\omega)x_1) + N_4(x; \alpha, \omega),\end{aligned}\tag{2.4}$$

cf. Lombardi (1996). Here $b_{1,2}$ are ω -dependent coefficients which must be determined for a particular example system and the remainder terms N_i , $i = 2, 3, 4$ are $\mathcal{O}(|x|^3 + \alpha|x|^2 + \alpha^2|x|)$. Moreover, assuming type-I reversibility (2.1) of the complete system we have that

$$N_2(Rx; \alpha, \omega) = N_2(x; \alpha, \omega), \quad N_3(Rx; \alpha, \omega) = -N_3(x; \alpha, \omega) \quad N_4(Rx; \alpha, \omega) = N_4(x; \alpha, \omega).$$

Note that these remainder terms will in general have an infinite Taylor series expansion and will contain terms that break the normal form structure. Hence the full system (2.4) will not be completely integrable.

A second-order truncated normal form is obtained by setting $N_i = 0$, $i = 2, 3, 4$. Of course, in order to write down this form of the truncated normal form, we have to assume that various co-efficients are non-zero. For example, systems with odd symmetry will not be expressible in the form (2.4) since the coefficients of the quadratic terms must be zero. In that case nonlinear terms and we would have to consider a third-order truncated normal form as the lowest order system.

In order to state the results that follow, Lombardi performs a further scaling for $\alpha = \lambda^2 > 0$, obtained by setting

$$x_1 = \lambda^2 y_1, \quad x_2 = \lambda^3 y_2, \quad x_3 = \lambda^2 y_3, \quad x_4 = \lambda^2 y_4, \quad t = \tilde{t}/\lambda,\tag{2.5}$$

Dropping (for the rest of this section only) the \sim on the new time variable, we obtain

$$\begin{aligned}\dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_1 - (3/2)y_1^2 - b_1(y_3^2 + y_4^2) + M_2(y; \lambda, \omega), \\ \dot{y}_3 &= -y_4((\omega/\lambda) + \lambda b_2 y_1) + M_3(y; \lambda, \omega), \\ \dot{y}_4 &= y_3((\omega/\lambda) + \lambda b_2 y_1) + M_4(y; \lambda, \omega).\end{aligned}\tag{2.6}$$

Here the M_i $i = 2, 3, 4$ are the obvious scalings of the remainder terms N_i which inherit their reversibility properties. Note that $M_2 = \mathcal{O}(\lambda^2)$ and $M_{3,4} = \mathcal{O}(\lambda^3)$ as $\lambda \rightarrow 0$.

Consider the truncated system obtained by setting $M_i = 0$, $i = 2, 3, 4$ in (2.6). This is again completely integrable with integrals J and K given by (2.3) where now explicitly

$$J = y_2^2 - y_1^2 + y_1^3 + 2dK^2 x_1.$$

Using these integrals it can be calculated that the truncated system has a one-parameter family of periodic orbits (up to phase shift τ)

$$v_K(t) := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} U(K) := \frac{1 - \sqrt{1 - 6K^2 d}}{3} \\ 0 \\ K \cos(\Omega_0(K)(t + \tau)) \\ K \sin(\Omega_0(K)(t + \tau)) \end{pmatrix},\tag{2.7}$$

in which

$$\Omega_0(K) = (\omega/\lambda) + b_2 \lambda U(K).$$

Moreover, the truncated system possesses a two-parameter family of homoclinic connections to the periodic solutions v_K given by

$$\gamma_{K,\tau}(t) := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_1^{(K)}(t) \\ y_2^{(K)}(t) \\ K \cos[\Omega_0(K)(t + \tau) + 4b\lambda V(K)\tanh(V(K)t)] \\ K \sin[\Omega_0(K)(t + \tau) + 4b\lambda V(K)\tanh(V(K)t)] \end{pmatrix}.\tag{2.8}$$

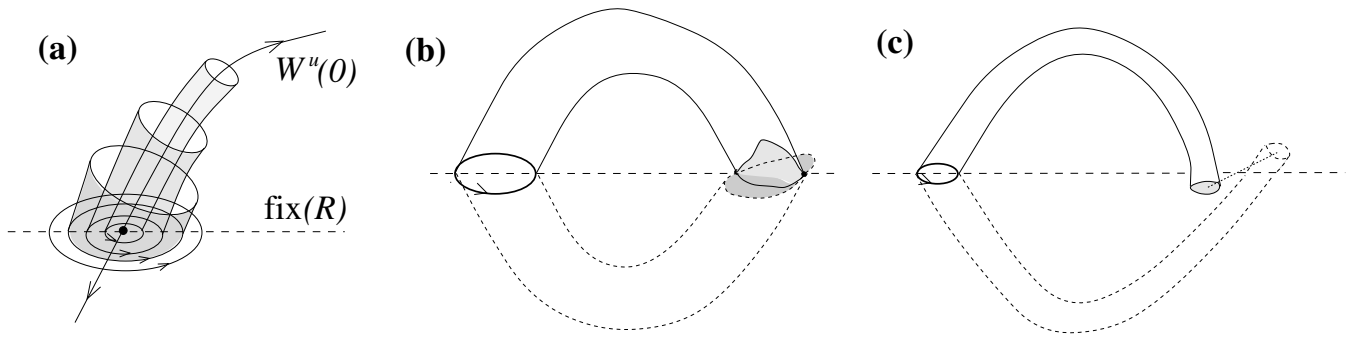


Figure 3: A schematic illustration of the existence of homoclinic solutions to periodic orbits in a neighbourhood of a perturbed singular saddle centre for $\lambda > 0$: (a) a three-dimensional representation of the four-dimensional phase space near a saddle-centre equilibrium, in which only the unstable manifolds of the origin and periodic orbits are depicted; also the two-dimensional $\text{fix}(R)$ is depicted as one-dimensional; (b) the unstable manifold (bounded by solid curves) of a sufficiently large amplitude periodic orbit ($K > K_{\min}$) intersects $\text{fix}(R)$ in two places, corresponding to two reversible homoclinic orbits to the periodic orbit; (c) for a small enough periodic orbit ($K < K_{\min}$), the splitting of the unstable manifold of the equilibrium $W^u(0)$ from $\text{fix}(R)$ is sufficiently large to mean that there are no homoclinic connections to this orbit.

Here

$$\begin{aligned} V(K) &= (1/2)\sqrt{1 - 3U(K)} = (1/2)(1 - 6K^2d)^{1/4}, \\ y_1^{(K)} &= U(K) + 4V^2(K)\text{sech}^2(V(K)t), \\ y_2^{(K)} &= -8V^3(K)\text{sech}^2(V(K)t)\tanh(V(K)t). \end{aligned}$$

Note that, for each K^2 , there are two phase shifts τ that correspond to *reversible* homoclinic connections, namely $\gamma_{K,0}$ and $\gamma_{K,\pi/\Omega_0(K)}$. Finally, note that the truncated system also admits the unique homoclinic orbit to the origin

$$\gamma_0(t) := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} \text{sech}^2(t/2) \\ -\tanh(t/2)\text{sech}^2(t/2) \\ 0 \\ 0 \end{pmatrix}. \quad (2.9)$$

A rigorous argument in Lombardi (1996, 1997b) shows firstly that a periodic solution $v_{K,\lambda}$ close to each v_K persists for small $\lambda > 0$ when the remainder terms M_i , $i = 2, 3, 4$ are included in (2.6). Secondly, for $K > K_{\min}$ where K_{\min} is an exponentially small function of λ , specifically

$$K_{\max} > C\lambda^2 \exp(-\omega\ell\pi/\lambda) \quad \text{for all } 0 < \ell < \pi \quad \text{and some constant } C > 0,$$

the pair of *reversible* homoclinic orbits $\gamma_{K,0}$, $\gamma_{K,\pi/\Omega_0(K)}$ also persist. See Figure 3 for a qualitative illustration. However, the question of the persistence of the homoclinic orbit to 0, i.e whether we can extend the existence down to $K = 0$, is much more delicate as we shall now see.

2.2 Persistence of homoclinic orbit to the origin

According to the discussion at the start of this section on the codimension of homoclinic connections, if a homoclinic orbit is to occur for $\lambda > 0$ then the least degenerate way is for $W^s(0)$ to intersect the symmetric section $\text{fix}(R)$. Even such connections are of codimension one and so, for fixed ω , we should *not* expect homoclinic orbits to 0 to persist for an interval of λ values as was the case for homoclinic orbits to a periodic orbit of a given finite size. Instead, we may at best expect to see homoclinic orbits along curves in the (λ, ω) -plane, as was effectively the case in Figure 1. The question we therefore wish to address using the normal form theory is how and at what ω -values such curves may bifurcate

at small amplitude from the limit $\lambda = 0$. The argument we shall adopt follows from calculations in Lombardi (1999, 1997a), to which the reader is referred for the technical details.

Consider, for the full system (2.6), the branch $W_+^s(0)$ of the stable manifold that corresponds to the homoclinic orbit of the unperturbed system. Let $y^s(t)$ be a solution in this manifold which we write in the form

$$y^s(t) = \gamma_0(t) + w(t), \quad (2.10)$$

where $\gamma_0(0) \in \text{fix}(R)$.

Now suppose we write (2.6) in the form

$$\dot{y} = F(y; \lambda, \omega) + \rho M(y; \lambda, \omega) \quad (2.11)$$

where $y = (y_1, y_2, y_3, y_4)^T$, F is the vector field of the truncated normal form and $M = (0, M_2, M_3, M_4)^T$ is the vector containing the remainders. Moreover $\rho \in [0, 1]$ is an *artificially introduced* homotopy parameter allowing one to pass from the truncated to the full system. A straightforward calculation then shows that the differential equation satisfied by the remainder term $w(t)$ in (2.10) is

$$\dot{w} - DF(\gamma_0(t)) \cdot w = S(w, t; \lambda, \omega) \quad (2.12)$$

where the nonlinear term is given by

$$S(w, t; \lambda, \omega) = F(\gamma_0(t) + w) - F(\gamma_0(t)) - DF(\gamma_0(t)) \cdot w + \rho M(\gamma_0(t) + w) \cdot w.$$

For a homoclinic orbit we require $w(0) \in \text{fix}(R)$. Using the linearisation about the explicitly known homoclinic solution of γ_0 for $\rho = 0$, and the differential equation (2.12) it is possible to turn this condition into a Melnikov integral condition. Skipping the details (Lombardi 1999), this condition can be expressed as

$$I(\lambda, \omega) = \int_0^\infty \langle r_-(t), S(w, t; \lambda, \omega) \rangle dt = 0, \quad (2.13)$$

where $r_-(t)$ is a solution of the linearisation about the homoclinic solution that satisfies $r_-(0) = (0, 0, 0, 1)$. Geometrically, we may view (2.13) as requiring that w has no component in the direction which is anti-symmetric under R at the point at $t = 0$. From the form of $\gamma_0(t)$ and F_λ we obtain

$$r_-(t) = (0, 0, -\sin \psi(t), \cos \psi(t)),$$

where

$$\psi(t) = (\omega/\lambda + a\lambda)t + 2b\lambda \tanh(t/2).$$

Despite being almost expressible in closed form, the Melnikov integral $I(\lambda)$ is not much use because it is currently expressed in terms of the unknown function $w(t)$. However, by using complex variable methods Lombardi (1999) is able to show that, for small ρ , $I(\lambda)$ is dominated by its w -independent part, that is

$$I(\lambda, \omega) = I_1(\lambda, \omega)(1 + \mathcal{O}(\rho)) \quad (2.14)$$

where

$$\begin{aligned} I_1 &= \rho \int_0^\infty \langle r_-(t), M(\gamma_0(t); \lambda, \omega) \rangle dt \\ &= \rho \int_0^\infty [M_4(\gamma_0(t); \lambda, \omega) \cos \psi(t) - M_3(\gamma_0(t); \lambda, \omega) \sin \psi(t)] dt. \end{aligned} \quad (2.15)$$

$I_1(\lambda, \omega)$ and more generally $I(\lambda, \omega)$ is termed an *oscillatory integral*.

For small ρ , we have thus reduced the existence of homoclinic solutions for non-zero λ to the vanishing of the closed form integral expression (2.15). This integral does *not* depend on the remainder M_2 . Moreover, owing to the form of $\gamma_0(t)$, when $M_{3,4}$ are expressed in Taylor series, it is only the terms

which are pure monomials in y_1 and y_2 which will contribute to I_1 . So we seem to have made progress. However there are some drawbacks. since even if M were known explicitly for a particular example, the analytical evaluation of the oscillatory integral is a complex task. However, one very important property of the evaluation of these integrals may be gleaned as we shall now see. The significance of this property appears only when I_1 is considered to be a function of both λ and ω .

2.3 Zeros of Bessel functions

In fact, by a series expansion of the remainder terms M , Lombardi (1999) shows that for small λ , I_1 satisfies

$$I_1(\lambda) = \frac{\rho}{\lambda^2} \exp(-\omega\pi/\lambda)(\Lambda(M_3, M_4, \omega) + \mathcal{O}(\lambda)) \quad (2.16)$$

where $\Lambda(M_3, M_4, \omega)$ can be computed explicitly for each monomial pure in y_1 and y_2 in the Taylor series expansion of either M_3 or M_4 . In fact, for each monomial, Λ is a (single signed ω -dependent) constant multiple of either a Bessel function $\Lambda \propto \mathcal{J}_n(4\sqrt{b_2\omega})$ for some integer n , for $b_2 > 0$; or a modified Bessel function $\Lambda \propto I_n(4\sqrt{b_2\omega})$ for $b_2 < 0$. Specifically, upon writing a pure monomial term as

$$M_i = kx_1^{m_1}x_2^{m_2}, \quad i = 3 \text{ or } 4;$$

then, for $b_2 > 0$,

$$\Lambda = k(-1)^{\text{int}(\frac{2m_1+m_2-1}{2})} \left(\frac{\omega}{b_2}\right)^{2m_1+3m_2-1} \mathcal{J}_{2m_1+3m_2-1}(4\sqrt{b_2\omega}), \quad (2.17)$$

where $\text{int}(\cdot)$ represents the integer part; and, for $b_2 < 0$,

$$\Lambda = k(-1)^{\text{int}(\frac{2m_1+m_2-1}{2})} \left(\frac{-\omega}{b_2}\right)^{2m_1+3m_2-1} \mathcal{I}_{2m_1+3m_2-1}(4\sqrt{-b_2\omega}).$$

Recall the basic properties of Bessel functions (Abramowitz & Stegun 1964) that $\mathcal{J}_n(x)$ has infinitely many zeros for $x > 0$ whereas the modified Bessel function $\mathcal{I}_n(x) = i^{-n}\mathcal{J}_n(ix)$ has no zeros.

Notice then the crucial role played by the coefficient b_2 in the truncated, scaled normal form (2.6) in the case that the remainder M consists of a single monomial in (y_1, y_2) . If $b_2 > 0$ there will be an infinite number of ω -values corresponding to zeros of Λ and hence homoclinic solutions will exist for small ρ and λ . However, if $b_2 < 0$ then Λ is strictly non-zero and hence there can be no homoclinic solution to the origin.

For a particular application, the remainder terms $M_{3,4}$ may well have an infinite Taylor series expansion. Hence the evaluation of precisely which values of b_2 and ω lead to zeros is non-trivial. Nevertheless, since zeros of Bessel functions are isolated, one has *generic* non-persistence for small ρ . Specifically (Lombardi 1999) proves that, for small ρ and all but a countable set of (explicitly calculable from (2.16)) values of ω , $b_{1,2}$ and remainders $M_{3,4}$, then $I_1(\lambda, \omega) \neq 0$ for all λ sufficiently small.

The biggest drawback is that ρ is an artificial parameter. In general applications one cannot necessarily put a small parameter in front of the terms that perturb the truncated normal form. Setting $\rho = 1$ does not allow the estimate (2.14), and so $I(\lambda)$ cannot be expressed in closed form. By a lengthy estimation of the integrals involved using complex variable theory, Lombardi (1997a) can show that *generic* choices of M , b, c, d and ω do not lead to homoclinic connections to the origin for sufficiently small λ . At first sight, this does not seem to be an improvement on the statement that homoclinic orbits to saddle-centres are codimension one.

However, recall that after the scaling (2.5) leading to the normal form (2.6), the remainder terms $M_{3,4}$ are $\mathcal{O}(\lambda^2)$. So, for an example system we should typically expect to find that its unscaled normal form (2.6) has remainder terms $N_{3,4}$ that are $\mathcal{O}(1)$ as $\lambda \rightarrow 0$. Hence, if we think of ρ in (2.11) as a scaling to make $M_{3,4}$ also $\mathcal{O}(1)$, then ρ may indeed be regarded as a small parameter, albeit not independent of λ ; $\rho = \mathcal{O}(\lambda^3)$. Hence in applications, for small λ , $I(\lambda, \omega)$ is likely to be well approximated by

$I_1(\lambda, \omega)$. Thus the zeros of the latter, explicitly calculable integral, may well yield a good indication of the ω -values at which homoclinic orbits bifurcate from $\lambda = 0$.

Let us now ponder on the result for a single monomial (equations (2.16), (2.17)) in the case $b_2 > 0$. Although for a given example one cannot say *a priori* which admissible monomial in the remainder term will contribute the most to I_1 , we do note that *each* such monomial contributes a term which is proportional to a Bessel function of the first kind. Moreover although the order of the Bessel function depends on the particular monomial, the argument of the Bessel function is always $4\sqrt{b_2}\omega$. From the asymptotics Bessel functions, in the limit of large $b_2\omega$ then *each* term contributing to I_1 will be approximately a sine wave in $4\sqrt{b_2}\omega$ with period $\approx 2\pi$. Now, suppose that one could take this limit of large $b_2\omega$ uniformly over all monomials. Then one has the (potentially infinite) sum of Bessel functions of different orders, and different amplitudes, but with the same argument. Hence in this large ω limit one would have the sum of sine waves in $\sqrt{b_2}\omega$, each with same period $\pi/2$. This sum itself would thus be a periodic function of the same period, and hence zeros of I_1 would be equally spaced in $\sqrt{b_2}\omega$ in the limit that this quantity gets large (recall that $b_2 = b_2(\omega)$). In particular, letting ω_n represent the n^{th} zero of $I_1(\omega)$ for fixed small λ , then in the limit as $n \rightarrow \infty$ one would have

$$\sqrt{\omega_{n+1}b_2(\omega_{n+1})} - \sqrt{\omega_nb_2(\omega_n)} \rightarrow \pi/4. \quad (2.18)$$

A similar reasoning would hold for the case $b_2 < 0$ if the limit $|b_2\omega| \rightarrow \infty$ could be taken uniformly for all contributions to I_1 corresponding to different monomials. Using the asymptotics of modified Bessel functions, in the limit we would obtain a sum of growing exponential functions with the same growth rate. For $b_2\omega$ sufficiently large then, the sum would also be a growing exponential function and we would have no zeros of I_1 and no persistence of homoclinic solutions.

The positive thing about the formula (2.18) for $b_2 > 0$ is that it gives the large ω asymptotics of the zeros of $I(\cdot, \omega)$ and quite remarkably depends only on the computation of $b_2(\omega)$ which is relatively straightforward for particular examples. The downside is that the reasoning that leads to its derivation requires a large number of assumptions that are difficult (if not impossible) to check for example systems. In order to test the validity of (2.18), numerical experimentation will now be carried out on a class of example systems to see whether it is at least right qualitatively.

3 Numerical experiments on a class of fourth-order equations

Consider the fourth-order ODE

$$y^{iv} + (\omega^2 - \lambda^2)y'' - \lambda^2\omega^2y + Ay^2 + Byy'' + Cy'''^2 + Ey'^2 + Gy'y''' + Ly'''^2 = 0. \quad (3.1)$$

This equation has the same form as (1.6) but the parameters have been redefined in such a way as to be precisely the moduli of the real and imaginary eigenvalues of the saddle centre equilibrium $y = 0$. Moreover the nonlinearity of (3.1) contains every possible quadratic term consistent with the system being reversible under (1.5).

3.1 Calculation of coefficient b_2

First let us rewrite this equation as a fourth-order system and put it into the normal form (2.4). Setting $\lambda = 0$, the linear substitution

$$(x_1, x_2, x_3, x_4) = (y + y'', y' - \omega y''', -\omega^2 y_3, +\omega^3 y_4) \quad (3.2)$$

puts equation (3.1) into the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{A}{\omega^2}x_1^2 - \frac{E}{\omega^2}x_2^2 + (B - 2\frac{A}{\omega^2})x_1x_3 + (C\omega^2 - \frac{A}{\omega^2} - B)x_3^2 \end{aligned}$$

$$\begin{aligned}
& + (2\frac{E}{\omega} - G\omega)x_2x_4 + (G\omega^2 - E + L\omega^4)x_4^2 \\
\dot{x}_3 &= -\omega x_4 \\
\dot{x}_4 &= \omega x_3 + \frac{A}{\omega^3}x_1^2 + -\frac{E}{\omega^3}x_2^2 + (\frac{B}{\omega} - 2\frac{A}{\omega^3})x_1x_3 + (\frac{B}{\omega} - \frac{A}{\omega^3} - C\omega)x_3^2 \\
& + (G\omega - \frac{E}{\omega} - L\omega^2)x_4^2 + (2\frac{E}{\omega^2} - G)x_2x_4.
\end{aligned} \tag{3.3}$$

We must next perform a scaling to make the coefficient of x_1^2 in the second equation equal to $-\frac{3}{2}$. This is achieved by defining new variables

$$\tilde{x}_i = \sigma x_i \quad \text{for } i = 1, 2, 3, 4 \quad \text{where } \sigma = \frac{2A}{3\omega^2}. \tag{3.4}$$

After neglection of the tildes on the new variables, this has the effect of scaling all the nonlinear terms in (3.3) by a factor of $1/\sigma$. In order to apply the preceding theory, all that remains is to calculate the coefficient b_2 in the parameter-independent normal form. Note from the structure of the normal form that among all possible quadratic transformations that preserve the reversibility (1.5), only those of the form

$$x \rightarrow x + \mathcal{G}(x) \quad \text{where } \mathcal{G} = \begin{pmatrix} 0 \\ 0 \\ g_1x_1x_3 + g_2x_2x_4 \\ g_3x_1x_4 + g_4x_2x_3 \end{pmatrix}$$

can affect the coefficient b_2 . Standard normal form theory (Kuznetsov 1995) shows that the effect of such a transformation to quadratic order is to add a terms $D\mathcal{G}\mathcal{F} - D\mathcal{F}\mathcal{G}$ where \mathcal{F} is the linear part of the vector field. Setting the transformed vector field equal to the appropriate quadratic part of the normal form leads to

$$\begin{pmatrix} 0 \\ 0 \\ (g_1 + \omega(g_4 + g_2))x_2x_3 + \omega(g_3 - g_1)x_1x_4 \\ (g_3 - \omega(g_4 + g_2))x_2x_4 + \omega(g_3 - g_1)x_1x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -b_2x_1x_4 \\ -q_3x_2x_4 + (b_2 - q_4)x_1x_3 \end{pmatrix}, \tag{3.5}$$

where

$$q_3 = 2\frac{E}{\omega^2\sigma} - \frac{G}{\sigma} \quad \text{and} \quad q_4 = \frac{B}{\omega\sigma} - 2\frac{A}{\omega^3\sigma}$$

are the appropriate quadratic coefficients of the linearly transformed vector field (3.3).

Equating coefficients of the the system (3.5) we find simply that

$$b_2 = \frac{q_4}{2} = -\frac{3}{4\omega} + 3\frac{B\omega}{2A} \tag{3.6}$$

This expression for b_2 now enables to us to predict, using (2.18) in the limit of large ω , that the zeros of $I(\cdot, \omega)$ for $B/A > 0$ will be periodic in ω with period given by

$$\Delta = \lim_{n \rightarrow \infty} \omega_{n+1} - \omega_n = \sqrt{\frac{2A}{3B}} \frac{\pi}{4} \tag{3.7}$$

Notice that the only coefficients of the quadratic terms in the original equation that (3.6) and (3.7) depend on is the ratio of B to A . Moreover by setting $y \rightarrow y/A$ then we can re-scale (3.1) so that $A = 1$ without loss of generality. The numerical experiments that follow are designed to test the inference from (3.6) that $B > 0$ is the condition for codimension-one persistence and that the bifurcation values of ω are eventually periodic in ω with period Δ .

3.2 Numerical procedure

The numerical test is the following, to use a high-accuracy numerical integrator (NAG routine `d02cjf`) to compute the positive component $W_+^u(0)$ of the unstable manifold of the origin of (3.1) and seek solutions which satisfy the reversibility condition $y' = y''' = 0$ at some x -value. An approximation to W_+^u is obtained by choosing an initial condition

$$(y(0), y'(0), y''(0), y'''(0)) = \varepsilon \mathbf{v}_u(\lambda) = \varepsilon \frac{(1, \lambda, \lambda^2, \lambda^3)}{\sqrt{1 + \lambda^2 + \lambda^4 + \lambda^6}}.$$

Here $\mathbf{v}_u(\lambda)$ is a unit eigenvector corresponding to eigenvalue $+\lambda$ and ε is a small positive number, $\varepsilon = 10^{-6}$ being found to be small enough in practice. By standard unstable manifold results, the error in making this approximation will be $\mathcal{O}(\varepsilon^2)$. The equation (3.1) is then solved forwards in x to produce a solution $y_u(x)$. The computation is stopped at the first value $x^* > 0$ for which $y'_u(x^*) = 0$. At this x -value which is located accurately by the integrator, the value of y''' is recorded. If it is zero then to, within numerical accuracy, a reversible homoclinic orbit will have been detected.

Owing to the estimate (2.16), this procedure was carried out for *fixed* λ but with ω allowed to vary, leading to a numerically defined function

$$T_\lambda(\omega) = \exp(\pi\omega/\lambda)y'''_u(x^*). \quad (3.8)$$

Now, by standard theorems for initial value problems, T_λ will be a smooth function of ω provided the *first* zero of y' remains well defined for all ω , that is provided there are no discontinuous jumps in x^* as ω is varied. The graphs of solutions in Figure 1 however, suggest that this property may not be true. However, in all the numerical runs for the system (3.1) we tried, provided λ was not too large (e.g. $\lambda < 2.0$) it was found that this jumping did not occur (but see figure 7(c) below for a case with cubic nonlinearity). Specifically, the first part of all trajectories either had the beginnings of the characteristic sech^2 shape with its well-defined maximum as the first extremum of $y(x)$ (see Figure 4(d) below), or the solution blew up to infinity without reaching an extremum. Since in such cases T_λ is a smooth function of ω , good numerical evidence for the existence of a homoclinic orbit is a sign change in $T_\lambda(\omega)$.

Performing numerics with $\lambda > 0$ fixed has the advantage of avoiding the singular limit altogether which, as already mentioned in the caption to Figure 1, leads to numerical difficulties. Nevertheless, for ω large, T_λ also becomes unreliable numerically because $y'''_u(x^*)$ becomes of the same order as the numerical accuracy. This can lead to spurious detection of zeros. However, as a signature of this behaviour the numerically determined $T_\lambda(\omega)$ was found to become non-smooth. So, computation of T_λ was only performed up to some ω_{\max} for which the curve was smooth and repeatable under change of numerical parameters. Taking λ larger typically meant that ω_{\max} could be increased.

Another strategy tried was to compute the formula (3.8) as a function of ω for fixed λ/ω . This has the advantage that computations can be performed up to larger ω -values, but the disadvantage that at those values λ is also large so one can be less sure that the numerics really describe the singular limit. Also, x^* varies greatly with ω in this case which can lead to numerical discretisation problems. Finally, as we shall shortly, the exponential scaling suggested by (2.16) does not appear to hold for all nonlinearities in (3.1). On balance it was found to be more reliable to keep λ fixed and to compute $T_\lambda(\omega)$ for $0 < \omega < \omega_{\max}$.

A balance was found for most of the numerical results that follow by taking $\lambda = 1$ (occasionally $\lambda = 0.75$) and $\omega_{\max} = 7\pi/2$. This value of λ may seem large when our purpose is to investigate the singular limit $\lambda \rightarrow 0$. Nevertheless, it was found that homoclinic orbits, if they occur, lie approximately on lines $\omega = \text{const.}$ up to large values of λ (see Figure 4(c) below). To check that we were indeed computing curves that really do bifurcate from $\lambda = 0$, computations of $T_\lambda(\omega)$ were also performed by systematically varying λ for certain of the nonlinear terms (as in Figures 4(c) and 6 below).

B	C	E	G	L	N_z	p/n/b/f	ω_{n-1}	ω_n	ω^{diff}	Δ
$\sqrt{5/6}$	0	$\sqrt{5/6}$	0	0	20	p	10.102	10.630	0.528	0.671
1/2	0	1/2	0	0	13	p	10.067	10.828	0.761	0.907
1/2	0	0	0	0	11	p	9.929	10.777	0.848	0.907
1/2	0	-1/2	0	0	9	p	9.292	10.222	0.920	0.907
1	0	0	0	0	18	p	10.333	10.868	0.535	0.641
1	-1	0	0	0	17	p	9.846	10.426	0.580	0.641
1	1/2	0	0	0	17	p	9.078	8.609	0.469	0.641
1	1	0	0	0	0	n	—	—	∞	0.641
1	0	0	-1	0	2	b (3.6)*	1.881	2.313	0.432	0.641
1	0	0	-1/4	0	17	p	9.546	10.124	0.578	0.641
1	0	0	1	0	0	n	—	—	∞	0.641
1	0	0	1/4	0	9	p	5.275	5.836	0.561	0.641
1	0	0	0	1	0	b (0.0)*	—	—	∞	0.641
1	0	0	0	1/4	12	b (8.6)*	8.156	7.388	0.786	0.641
1	0	0	0	-1	0	n	—	—	∞	0.641
1	0	0	0	-1/4	8	f*	3.773	4.415	∞	0.641
-1/2	0	-1/2	0	0	0	b (0.0)	—	—	∞	∞
-1/4	0	-1/4	0	0	0	n	—	—	∞	∞
-1/2	0	0	0	0	0	n	—	—	∞	∞
-1/2	0	1/2	0	0	0	n	—	—	∞	∞
-1/2	0	2	0	0	6	p	8.314	10.442	2.128	∞
-1	0	1	0	0	1	f	—	—	∞	∞
-1/2	1/2	0	0	0	0	n	—	—	∞	∞
-1/2	-1/2	0	0	0	0	b (0.0)	—	—	∞	∞
-1/2	0	0	1/2	0	0	n	—	—	∞	∞
-1/2	0	0	-1/2	0	0	b (0.3)	—	—	∞	∞
-1/2	0	0	0	$\pm 1/2$	0	b (0.0)	—	—	∞	∞
0	0	0	0	0	0	n	—	—	∞	—
0	1	0	0	0	0	n	—	—	∞	—
0	-1	0	0	0	0	b (0.0)	—	—	∞	—
0	-1/2	0	0	0	7	p	9.091	10.403	1.314	—
0	0	1	0	0	8	p	9.188	10.431	1.253	—
0	0	0	1/2	0	0	n	—	—	∞	—
0	0	0	-1/4	0	4	b (10.0)*	5.560	6.729	1.169	—
0	0	0	0	1/4	0	b (1.0)	—	—	∞	—
0	0	0	0	-1/4	0	n	—	—	∞	—

Table 1: *Summary of persistence of homoclinic orbits for the fourth-order equation (3.1) with $A = 1$, $\lambda = 1$, $0 < \omega < 7\pi/2$ and the B , C , E , G and L values specified in the first five columns. N_z is then the number of zeros of $T_1(\omega)$ defined by (3.8) that were detected for ω in the given interval. The next column provides some qualitative information: p implies a sequence that is apparently **p**periodic in ω (at least for ω sufficiently large) as in Figure 4(a); n implies that $T_1(\omega)$ is well-defined and has **n**o zeros; b implies that for sufficiently large ω (given in brackets) solutions in the unstable manifold **b**low up in finite time before reaching a zero of y' , hence T_1 is undefined; and f corresponds to when the numerics suggest there are a **f**inite number of zeros, i.e. there are no homoclinic solutions for ω sufficiently large, but the function $T_1(\omega)$ is well defined; finally, an asterisk implies that by taking λ smaller there was evidence of a periodic sequence of zeros of $T_\lambda(\omega)$. The columns headed ω_n and ω_{n-1} give the last two zeros that are not badly affected by error (i.e. for which T_1 is still smooth), and the next column computes their difference. The final column gives the theoretical value of this limit according to formula (3.7).*

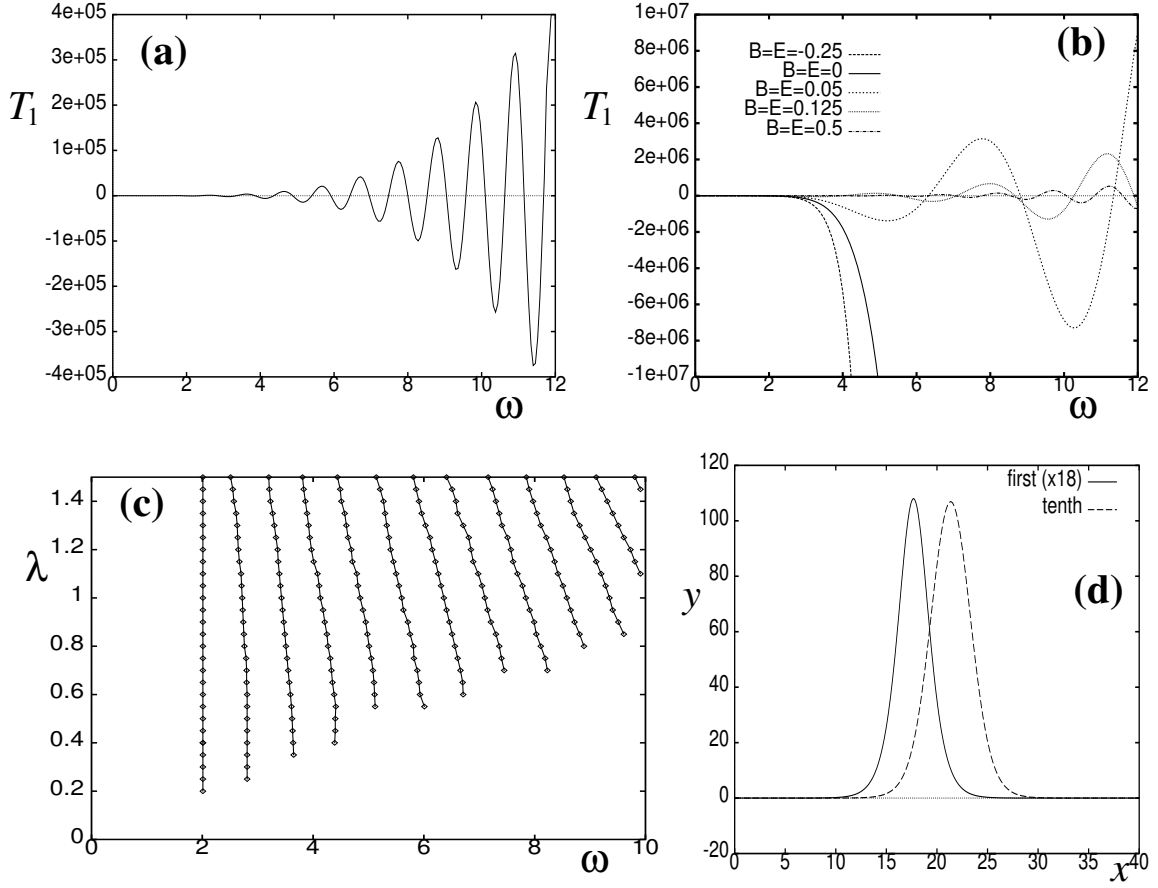


Figure 4: The function $T_1(\omega)$ for $A = 1$ and $B = E$ with (a) $B = \sqrt{5/6}$ and (b) a range of different values of B . In (c), for the case $B = E = 0.5$, the locus of zeros of $T_\lambda(\omega)$ is depicted. (d) Depicts the solution $y(x)$ corresponding to the first and tenth zeros of $T_1(\omega)$. The first solution is multiplied by a factor of 18 for comparison purposes.

3.3 Results

Table 1 presents a summary of the results of the computation of $T_1(\omega)$ for (3.1) with $A = 1$ and a variety of combinations of the other nonlinear terms.

Note that Equation (1.8), after the appropriate scaling of y and x , is equivalent to (3.1) with $A = 1$, $B = E = \sqrt{5/6}$ and all other nonlinear coefficients zero. The first line in Table 1 summarizes this case, and the corresponding curve $T_1(\omega)$ is depicted in Figure 4(a). This confirms the findings in Champneys & Groves (1997) that there are regular sequence of persistence points. Figure 4(b) depicts $T_1(\omega)$ for other cases with $B = E$. Observe that when $B > 0$ then regular zeros occur, and that for $B \leq 0$ the curve T_1 has no zeros. In fact, as indicated in the first row of the second block of data in Table 1, if B is sufficiently negative then T_1 becomes undefined because solutions $y(x)$ with initial conditions in the unstable manifold of the origin blow up without at a finite x -value. This then is in agreement with the theoretical result (3.6), (3.7) that the sign of B for ω sufficiently large determines whether an approximately periodic sequence of persistence points occur, or none at all.

Before leaving the case with $B = E$, let us look at the quantitative information on the spacing of the persistence points for $B > 0$. The last two columns of the first two rows of Table 1 show acceptable quantitative agreement between the numerical spacing of the zeros of T_1 and the formula (3.7). (It should of course be remembered when comparing these quantities that the theoretical Δ is obtained in the limit as $\omega \rightarrow \infty$ and $\lambda \rightarrow 0$, but the numerics has $\lambda = 1$ and $\omega < 7\pi/2$.) For both $B = \sqrt{5/6}$ and $B = 1/2$ the numerics gives an underestimate of the limiting theoretical value which can be justified

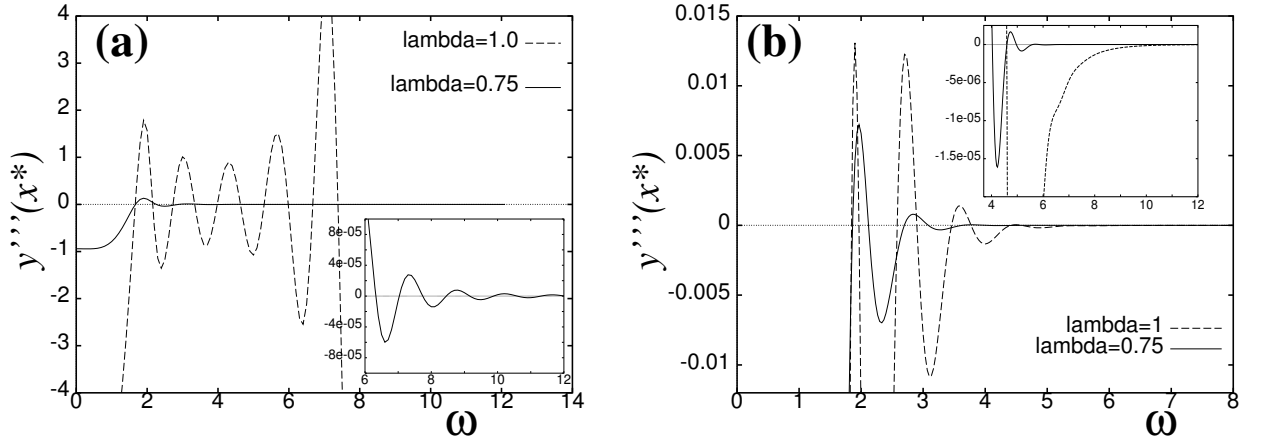


Figure 5: The function $y'''(x^*)$ against ω for $A = B = 1$, $C = E = H = 0$ and (a) $L = 0.25$ and (b) $L = -0.25$. In both cases, the curves for $\lambda = 1$ and $\lambda = 0.75$ are compared.

by noting that the zeros of Bessel functions for small arguments are more spaced out than in the large asymptotic limit. Finally note that the ratio between the data for the two B -values is 0.7 to one decimal place for both the numerical and theoretical values.

To show that this behaviour persists for different values of λ , Figure 4(c) shows the root locus of zeros of $T_\lambda(\omega)$ for the case with $B = E = 1/2$. Note that zeros, hence homoclinic solutions persist along lines given approximately by $\lambda = \text{const.}$, as predicted by the analysis. The ‘gap’ in the bottom right of the diagram and also for $\lambda < 0.2$ is where the numerics is unreliable for the reasons given in the previous subsection; that $y'''(x^*)$ becomes the size of numerical error and hence $T_\lambda(\omega)$ is non-smooth. We thus claim that the data presented is symptomatic of curves which reach all the way down to $\lambda = 0$, although we cannot prove it. This then illustrates the delicacy of trying to use numerics to probe the singular limit; $\lambda = 1$ is chosen as a trade-off between detecting sufficiently many zeros reliably and having the qualitatively correct description of the singular limit. Finally for the case $B = E = 1/2$, Figure 4(d) shows two homoclinic solutions corresponding to the first and tenth zeros of T_1 . Note that the graphs look qualitatively identical; $y(x)$ has a sech^2 -like profile. In particular, no extra oscillations superimposed onto the graphs can be discerned. Yet, if λ were taken to be even larger and these curves continued it is to be expected that, as in Figure 1 the solution corresponding to the n^{th} persistence point would develop $2n + 1$ internal zeros of $y'(x)$. We conjecture that the process by which these super-imposed oscillations form is exponentially small in λ/ω . (CAN I DO THIS?)

The rest of the first block of data of Table 1 shows other cases where B is positive. In most cases good agreement is found with the theory, namely that there is an approximately periodic sequence of zeros of $T_1(\omega)$ whose period is comparable with the theoretical limiting value Δ . In several cases, e.g. when $B = 1$ and $G = 1$ or $L = -1$, the data for T_1 is inconclusive since solutions blow up. However, decreasing λ reveals further evidence of a periodic sequence. Such evidence is presented in Figure 5 for two cases with $B = 1$ and $L = |1/4|$. When L is positive (Figure 5(a)), T_1 blows up when $\omega \approx 8.6$. However, for $\lambda = 0.75$, the inset to the figure shows $T_{0.75}$ to have regularly spaced zeros up to ω_{max} . Note that we have chosen to depict $y'''(x_*)$ rather than T_λ since, at least for $\lambda = 1$, T_1 does not appear to scale as suggested by (3.8). For L negative (Figure 5(b)) T_1 is oscillatory for small ω and then, as can be seen in the inset to that figure, remains finite but negative for $\omega > 4.415$. Taking $\lambda = 0.75$, however, shows that the sequence of zeros continues up to larger ω .

These cases where taking smaller λ leads to a periodic sequence are still in agreement with the theory, but there are some combinations of nonlinear coefficients which cannot be rectified with the theory. Three examples are given in the table: $B = C = 1$, $B = G = 1$ and $B = -L = 1$. In such cases we could find no direct evidence of regular sequences of zeros of T_λ for smaller λ . Also for $\lambda = 1$, there was no blow up of solutions for $\omega < \omega_{\text{max}}$.

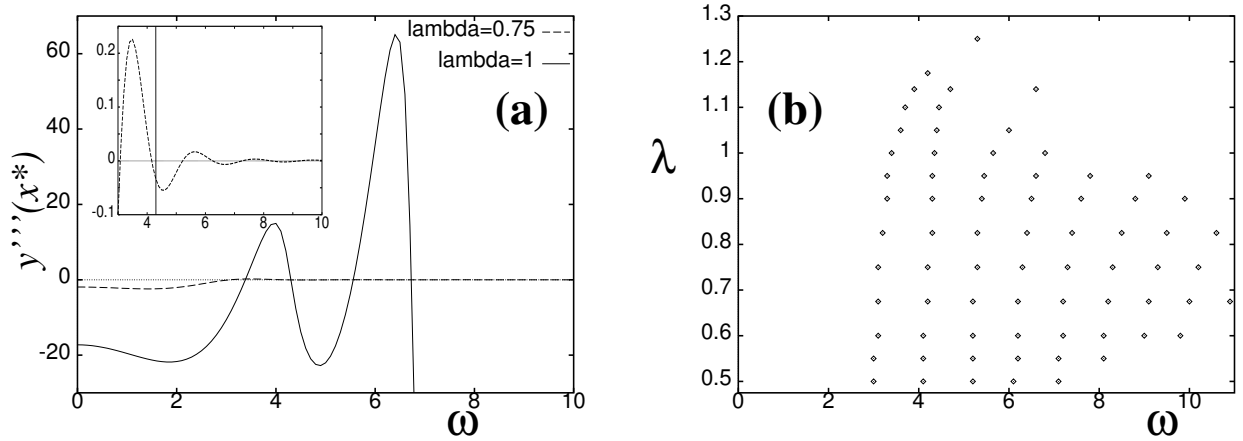


Figure 6: (a) The function $y'''(x^*)$ against ω for $A = 1$, $G = -0.25$ and $B = C = E = H = 0$, comparing the cases with $\lambda = 1$ and $\lambda = 0.75$. (b) The root locus of $T_\lambda(\omega)$ in the corresponding case.

The second block of data in Table 1 refers to cases with $B < 0$. Here the reasoning in Section 2.3 above would predicts no persistence points for ω large (this is what is meant by the symbol ‘ ∞ ’ in the final column.) In fact, in all but two cases presented in the table, it is found that T_1 has no zeros at all. The main contradiction to the theory appears to be the case with $B = -1/2$, $E = 2$ where a periodic sequence of zeros was detected, albeit with a much longer period than the cases for which $B > 0$. Note, as witnessed by the proceeding row in the table, that making $|B|$ larger and E smaller caused this periodic sequence to have longer period until only one zero was detected for $\omega < \omega_{\max}$.

The final block of data in Table 1 refers to examples where $B = 0$ and hence the large ω asymptotics according to (3.6) is $b_2 \sim 1/\omega$. Hence the argument in Section 2.3 does not give any information since one cannot pass to the limit of large $b_2\omega$. This is why a dash is entered in the final column of the table. Depending on the values of C , D , E , G and L , it was sometimes found that there were no zeros of T_1 or that (for $C = -1/2$ or $E = 1$) there was again an approximately periodic sequence (with period longer than cases with $B \geq 1/2$). The case with $G = -1/4$ is illustrated in Figure 6(a). Here T_1 has finitely many zeros up to $\omega \approx 10$, beyond which value solutions blow up before a zero of y' is reached. Taking $\lambda = 0.75$ however, leads to an apparently periodic sequence of zeros. Finally, Figure 6(b) shows zeros of T_λ in the (ω, λ) -plane. Note that the ‘gap’ region to the bottom and the right is due to $y'''(x^*)$ being the size of machine precision as it was in Figure 4(c), but that the gap to the upper right is due to blow up. There is some evidence in this figure to suggest that prior to blow up, loci of zeros disappear to $\omega = \infty$. Although it may just be that the homoclinic solutions go to infinity in L_2 -norm at finite ω -values, and hence loci end ‘in mid air’.

4 Discussion

Let us now comment on what the preceeding numerics do and do not show. The first obvious point is that they illustrate why the singular limit is delicate. Depending on coefficients of nonlinear terms, we either get a regular sequence of persistence points, or finitely many, or none at all. It would be fair to say, with a few caveats, that the sign of b_2 for large ω , is a good indication of which of these occurs, as suggested by the theory. Moreover, when persistence occurs then it appears at points which are approximately periodic in ω . This agrees with the formula (3.7) and comes about because b_2 is $\mathcal{O}(\omega)$ for the particular example (3.1) and the relevent quantity from the theory in section 2 is $\sqrt{b_2\omega}$. There is some quantitative agreement too in the period of these zeros.

In one sense it is remarkable that there is any agreement between the tentative interpretation of Lombardi’s thoery we made in section 2.3 and the numerics results. The former assumes that λ is sufficiently small and ω sufficiently large, which are precisely the limits that the numerics cannot

explore. Also, the theory relied on some assumptions about the effect of the infinite Taylor expansion of the remainder terms. It is not clear at present which of these assumptions causes the caveats in the numerical results, i.e. the cases where the calculated b_2 is positive (negative) but the numerics gives no (respectively a periodic sequence of) persistence points. The best conjecture at the moment would appear to be the λ -dependent terms in the computation of b_2 would have to be taken into account. For instance, if, instead of the linear transformation (3.2) which was based on eigenvectors at $\lambda = 0$, we make the substitution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda^2 - \lambda & \lambda + 1 & 1 & 0 \\ \lambda^2 + \lambda^3 & \lambda^2 - \lambda & 0 & -\omega \\ \lambda^4 - \lambda^3 & \lambda^3 + \lambda^2 & -\omega & 0 \\ \lambda^5 + \lambda^4 & \lambda^4 - \lambda^3 & 0 & \omega^3 \end{bmatrix} \begin{bmatrix} y \\ y' \\ y'' \\ y''' \end{bmatrix},$$

this puts the linear part of (3.1) in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Omega \\ 0 & 0 & \Omega & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

which is the same as for the parametrised normal (2.4). However, after performing this linear substitution, the coefficients of the nonlinear terms in (3.3) become more complicated. For example, q_4 defined by (3.5) now depends on A , B , λ and C , and σ defined by (3.4) now depends on λ and *each* of the nonlinear coefficients A , B , C , E , G and L . Hence for finite λ , b_2 is in effect a more complicated expression.

It should be stressed that the numerics have only considered a limited class of four-dimensional reversible systems, albeit well-motivated by extended 5th-order KdV examples, namely ones that can be expressed in the form (3.1). Future work will also look at systems of two second-order equations, such as that studied by (Grimshaw & Cook 1996). Also, we have not looked at the effect of terms which are of higher-order than quadratic. Such terms cannot affect the coefficients b_1 and b_2 of the second-order truncated normal form, but they can alter the structure of the remainder terms M_i . Nor have we treated the case which often occurs in applications, such as the nonlinear-optical model (1.9), where the system has odd symmetry so that necessarily $b_{1,2} = 0$ and cubic terms must be considered in the lowest order truncated normal form.

We shall leave the systematic study of systems with cubic and higher-order nonlinear terms for future work. However, Figure 7 presents some evidence to show how things may be either similar or very different from the purely quadratic case considered here. Figure 7(a) depicts the curve T_1 for the odd-symmetric model studied by Kivshar et al. (1998) where, like (1.8), the derivative-dependent nonlinear terms stem from a single term in the Hamiltonian function. For this example, the behaviour is very much like the pure quadratic nonlinearities studied in Section 2 above; there is a sequence of zeros that is approximately periodic in ω . However, Figure 7(b) which is for a mixed quadratic and cubic nonlinearity shows very different behaviour; there are many more persistence points (of the order of a hundred for $\omega < 4$), these appear to be periodic in ω^n for some $n > 1$. Finally Figure 7(c) which is for another mixed quadratic/cubic nonlinearity shows that $T_1(\omega)$ appears to be piecewise linear. The jumps can be explained by the fact that the graph of $y(x)$ is no longer sech^2 -like, and has oscillations superimposed (more obvious in the graph of $y''(x)$).

When looking to use the results of this paper in applications, we also need to consider the case where, despite having purely quadratic nonlinearity, the coefficient of b_2 may be precisely zero, a straightforward calculation shows this to be the case for the nonlinear-optical model (1.10). Also, recall that for the optics model (1.11) an as yet unexplained phenomenon is observed numerically; that a countable set of curves of homoclinic solutions appear to emanate from the doubly singular point where $\lambda = \omega = 0$ (see (Champneys & Malomed 1999a) for the details).

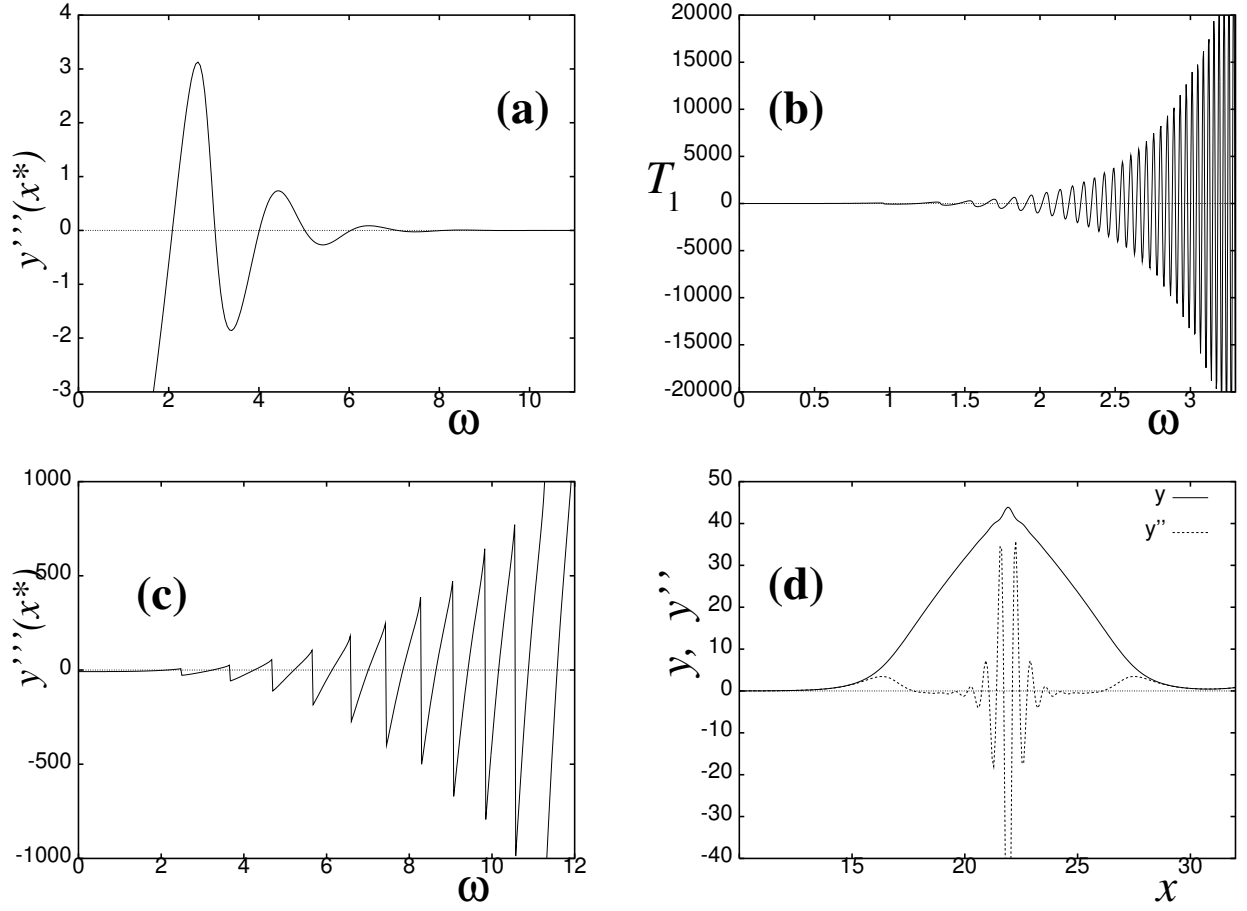


Figure 7: Illustrating $T_1(\omega)$ for models of the form (1.6) with g containing cubic terms: (a) $g = y^3 + (3/4)y(yy'' + y'^2)$; (b) $g = y^2 + y^2y''$; (c) $g = y^2 + yy'^2$. (d) shows the solution as $y(x)$ and $y'''(x)$ at the eighth actual zero, $\omega = 8.6482734061$ for case (c).

We also need to consider whether these results can be applied in a centre-manifold sense to problems of higher dimensions. Given reversible systems in more than four dimensions with four eigenvalues given by (1.2) and the rest of the spectrum off the real axis, it would be tempting to appeal to the centre-manifold theorem to say that one should expect to see codimension-one persistence as predicted by the asymptotics of the effective $b_2(\omega)$ for the four-dimensional sub-system. However, if there is other spectrum on the imaginary axis, then the codimension of homoclinic solutions will go up, as is the case in the example systems studied in Champneys & Malomed (1999*b*, 1999*a*) when one considers optical solitons with non-zero velocity. Finally can anything be said about the infinite dimensional case, such as the Euler-equation formulation of the water wave problem? Lombardi (1999) makes some comments on what would be required to apply his non-persistence result to the full water-wave problem, based on his success in so applying his results on persistence of homoclinics to periodics (Lombardi 1997*b*). He also quotes a computation in Amick & Kirchgassner (1989) that (at least in part of parameter space) b_2 is negative for water waves.

In summary, this paper has presented a simple criterion for estimating the persistence of solitary waves beyond all orders in a singular limit that occurs frequently in nonlinear-wave systems. This criterion has been shown to agree reasonably well with numerical experiments on a class of fourth-order equations. It should be stressed that we have not attempted to write a rigorous proof of persistence. For an arbitrary system, such a proof may be hard to obtain as it depends on estimating the exponentially small splitting function resulting from *every* term in the Taylor series expansion of the vector field and proving that as ω varies there must be zeros of this function. But as a ‘rule of thumb’ the positiveness of b_2 , which is relative easy to calculate as it involves a finite number of nonlinear transformations, appears from the numerics to provide the right answer in most cases. It is the hope of the author that this tentative conclusion may provide a stimulus to the rigorous mathematical community to seek more definite results on codimension-one persistence in example systems.

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